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Associators in the Nucleus of Antiflexible Rings

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Abstract

In this paper, first we prove that if R is a semi prime third power associative ring of char ≠ 2 then either N = C or R is associative. Using this result we prove that if R is a simple third power associative antiflexible ring of char ≠ 2,3 satisfying (x, x, y) = k (y, x, x) for all x,y ϵ R, k ≠ 0 and $3k^2$ + 2k + 1 ≠ 0 then either R is associative or nucleus equals center.

Keywords: *Associator, commutator, nucleus, center, simple ring, prime ring.*

Introduction

E. Kleinfeld and M. Kleinfeld [3] studied a class of Lie admissible rings. Also in [4] they have proved some results of a simple Lie admissible third power associative ring R satisfying an equation of the form (x,y,x) = k(x,x,y) for all x,y ϵ R, k ≠ 0,1 and k^2+2 ≠ 0. In this paper, we prove that if R is a simple third power associative antiflexible ring of char ≠ 2,3 satisfying (x, x, y) = k (y, x, x) for all x,y ϵ R, k ≠ 0 and $3k^2$ + 2k + 1 ≠ 0 then either R is associative or nucleus equals center.

Preliminaries

Let R be a non associative ring . We denote the commutator and the associator by (x,y) = xy – yx and (x,y,z) = (xy)z – x(yz) for all x,y,z ϵ R respectively. The nucleus N of a ring R is defined as N = { n ϵ R / (n,R,R) = (R,n,R) = (R,R,n) = 0}. The center C of ring R is defined as C = { c ϵ N / (c,R) = 0}.A ring R is called simple if R^2 ≠ 0 and the only non-zero ideal of R is itself. A ring R is called Prime if whenever A and B are ideals of R such that AB = 0, then either A = 0 (or) B = 0.

Main Results

Let R be an antiflexible, then it satisfies the identity

A(x, y, z) = (x, y, z) = (z, y, x) --------(1)

By the third power associativity, we have

(x, x, x) = 0 ---------(2)

Linearizing of (2) gives

B(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0 --------(3)

We use the following two identities Teichmuller and semi Jacobi which holds in all rings

C(w, x, y, z) = (wx, y, z) – (w, xy, z) + (w, x, yz) – w(x, y, z) – (w, x, y)z = 0 ------ (4)

And D(x, y, z) = (xy, z) – x(y, z) – (x, z)y – (x, y, z) – (z, x, y) + (x, z, y) = 0 -------(5)

We denote E (x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) -----(6)

Then from D(x, y, z) – D(y, x, z), we obtain

((x, y), z) + ((y, z), x) + ((z, x), y) = E (x, y, z) – E(x, z, y) -------(7)

As we observed by Maneri in [1], in any arbitrary ring with elements w, x, y, z we have

0 = C(w, x, y, z) – C(x, y, z, w) + C(y, z, w, x) – P(z, w, x, y)

= E(wx,y,z)–E(xy z,w)+ E(yz,w,x)–E(zw,x,y)–(w,(x,y,z))+(x,(y,z,w))–(y,(z,w,x)) +(z,(w,x, y)).

We now assume that R satisfies identity (3), E(a, b, c) = 0 for all a, b, c ϵ R.

So the above equation imply

(w, (x, y, z)) + (x, (y, z, w)) – (y, (z, w, x)) + (z, (w, x, y)) = 0 -----(8)

Let N be the nucleus of R and let n ϵ N. By substituting n for w in (8), we get

(n, (x, y, z)) = 0 ------(9)

i.e., n commutes with all associators.

The combination of (9) & (4) yields

$$(n, w(x, y, z)) = - (n, (w, x, y)z) \qquad \text{-----(10)}$$

If u and v are two associators in R, then substituting z=n, x=u, y=v in (5), we get

$$(uv, n) = 0 \qquad \text{----------(11)}$$

If u = (a, b, c) then ((a, b, c)v, n) = 0

Using (10), we have – (a(b, c, v),n) = 0

From this and (5), we obtain -a((b, c, v),n) – (a, n)(b, c, v) = 0

Using (9), we have (a, n) (b, c, v) = 0 -----(12)

Now we prove the following theorem.

Theorem: If R is semiprime third power associative ring of char $\neq 2$ which satisfies (3), then either N = C (or) R is associative.

Proof:

If N ≠ C, then there exist n ϵ N and a ϵ R such that (a, n) ≠ 0.

Hence from (12), we have (b, c, v) = 0 for all associators v and b, c ϵ R.

We can write this as (R, R, (R, R, R)) = 0.

By putting v = (q, r, s) in (11), we get (u(q, r, s), n) = 0

Using (10) this leads to –((u, q, r)s, n) = 0.

From this and (5), we obtain (u, q, r) (s, n) = 0.

Since N ≠ C , we have (u, q, r) = 0 for all associators u and q,r ϵ R.

We can write this as ((R, R, R), R, R) = 0.

Using (R, R, (R, R, R)) = 0 = ((R, R, R), R, R) the identity (3) gives

(R, (R, R, R), R) = 0

Thus (R,R,R) Ç N. Since R is semiprime, we use the result in [1] to conclude that R must be associative.

This completes the proof of the theorem.

Henceforth we assume that R satisfies an equation of the form

$$(x, x, y) = k (y, x, x) \qquad \text{------(13).}$$

for all x, y ϵ R, k ≠ 0 and using (3), identity (13) implies

(x, y, z) + (y, z, x) + (z, x, y) = 0

Putting y=x, z=y => (x, x, y) + (x, y, x) + (y, x, x) = 0

k(y, x, x) + (x, y, x) + (y, x, x) = 0

(k+1) (y, x, x) + (x, y, x) = 0

(x, y, x) = - (k+1) (y, x, x)

= - (k+1) (x, x, y)

$$= - \left(\frac{k+1}{k}\right) (x, x, y) \qquad (\text{ by (1) \& (13) }) \qquad \text{--------(14)}$$

Lemma: Let T = { t ϵ R / (t, N) =0 = (tR, N) = (Rt, N) }. Then T is an ideal of R.

Proof: Let t ϵ T, n ϵ N and x, y, z ϵ R.

Then (t.xy, n) = (t.xy, n) = 0 .

Using (9) and the definition of T. Also (5) implies (y.tx, n) = (y,n).tx.

But (5) also yields (yt, n) = (y, n)t = 0 since (yt,n) = 0.

Now, ((y, n), t, x) = ((y, n).t)x – (y, n).tx

= 0 – (y, n).tx

= - (y.tx, n)

or (y.tx, n) = - ((y, n), t, x) ---------(15)

Now consider , ((y, n), x, x) = (yn, x, x) – (ny, x, x) ------(16)

Using (14), (x, x, yn) = k (yn, x, x)

While 0 = C (x, x, y, n) = (x, x, yn) – (x, x, y)n and

0 = C (n, y, x, x) = (ny, x, x) – n(y, x, x)

Substitute this in (16) and using (14) & (9) gives

((y, n), x, x) = (yn, x, x) - (ny, x, x)

=(x, x, yn) – n(y, x, x) (by (1))

= (x, x, yn) – n(x, x, y) (by (1))

= (x, x, y)n – n(x, x, y) (by (4))

= ((x, x, y), n)

= 0

Linearizing the above identity, we get

((y, n), x, z) = - ((y, n), z, x) -----------(17)
Again consider ((x, n),y, x) = (xn, y, x) – (nx, y, x) ------(18)
From C (x, n, y, x) = 0, it follows that (xn, y, x) = (x, ny, x)
From (14), it follows that
(x, ny, x) = - (k+1) (x, x, ny) (by (14))
= - (k+1) (ny, x, x) (by (1))
And from C (n, y, x, x) = 0 we have
(ny, x, x) = n(y, x, x)
Thus we have (x, ny, x) = - (k+1) (x, x, ny) -----(19)
From C(n, x, y, x) = 0, it follows that (nx, y, x) = n(x, y, x)
While from (14), we have n(x, y, x) = - (k+1) n(x, x, y)
Therefore (nx, y, x) = n(x, y, x) = - (k+1) n(x, x, y) ------(20)
Substitute (19) & (20) in (18), we get
((x, n), y, x) = (xn, y, x) – (nx, y, x)
= - (k+1) n(x, x, y) + (k+1) n(x, x, y)
= 0 --------- (21)
Linearizing (21), we get
((x, n), y, z) = - ((z, n), y, x) -----(22)
Combining (17) and (22), we get ((π(x),n) , π(y), π(z)) = Sgn(π) ((x, n), y, z) -------(23)
for every permutation π on the set {x, y, z}.
Applying (23), we see that ((y, n), t, x) = ((y, n), x, t) (by (17))
= - ((t, n), x, y) (by (22))
= ((t, n), y, x) (by (17))
= 0 (by defn. of T)
Combined this with (15), we obtain (y.tx, n) = 0
So T is a right ideal of R. By using the anti-isomorphic ring, we similarly prove that T is a left ideal of R.
Therefore T is an ideal of R.

Theorem: If R is a simple third power associative antiflexible ring with (13) of char ≠ 2,3 is either associative or satisfies nucleus equals center, N = C .

Proof: Simplicity of R implies either that T = R or T = 0.
If T = R, then N = C.
Hence assume that T = 0.
Let u = (a, b, c) be an arbitrary associator with elements a, b, c ϵ R.
We have already observed that for every associator v, we have (uv, n) = 0.
Now using (C(u, x, x, y), n) = 0 and (9) gives
((u, x, x)y, n) = - (u(x, x, y), n) = 0
Using (C(y, x, x, u), n) = 0 gives (y(x, x, u), n) = - ((y, x, x)u, n) = 0
Also (14) implies that y(x, x, u) = k y(u, x, x)
⇨ $y(u, x, x) = \frac{1}{k} y(x, x, u)$
So (y(u, x, x),n) = 0 Since ((u, x, x), n) = 0 (by (7))
We have (u, x, x) ϵ T.
Since we are assuming T = 0, we have (u, x, x) = 0 for all x ϵ R.
Using this in (14), we get
(x, u, x) = 0 and (x, x, u) = 0
Thus (x, u, x) = (x, x, u) = (u, x, x) = 0 ----------(24)
For a, b ϵ R , we define a ≡ b if and only if (a-b, n) = 0 for all n ϵ N.
Let α = x(y, x, z)
Because of (9), all associators are congruent to zero.
Thus C (x, y, x, z) = 0 Implies α = - (x, y, x)z.
Equ (14) implies α = - (x, y, x)z = (k+1) (y, x, x)z
By using C(w, x, y, z) = 0 continuously and (14) yields
α = x(y, x, z)
≡ - (x, y, x)z
≡ $(\frac{k+1}{k})$ (x, x, y)z (by (14))

$\equiv$ - ($\frac{k+1}{k}$) x(x ,y ,z)
$\equiv$ ($\frac{k+1}{k}$) (x, x, y)z
$\equiv$ (k+1) (y, x, x)z
$\equiv$ - (k+1) y(x, x, z)
$\equiv$ k y(x, z, x)
$\equiv$ - k (y, x, z)x
$\equiv$ k y(x, z, x)
$\equiv$ - k (k+1) y(z, x, x)
$\equiv$ k (k+1) (y, z, x)x --------(25)

Permuting y and z in (25), we get
β = x(z, x, y) = - (x, z, x)y
= ($\frac{k+1}{k}$) (x, x, z)x
= - ($\frac{k+1}{k}$) x(x, z, x)
= ($\frac{k+1}{k}$) (x, x, z)y
= (k+1) (z, x, x)y
= - (k+1) z(x, x, y)
= k z(x, y, x)
= - k (z, x, y)x
= k z(x, y, x)
= - k (k+1) z(y, x, x)
= k (k+1) (z, y, x)x ----(26)

From identity (3) we obtain
x(x, y, z) + x(z, x, y) = - x(y, z, x) ----------(27)
using (25) and (26) in (27), we get
$(-\frac{k}{k+1})\alpha + \beta$ = - x(y, z, x) ----------(28)
However C (x, y, z, x) = 0 gives –x(y, z, x) $\equiv$ (x, y, z)x
Thus $(-\frac{k}{k+1})\alpha + \beta$ = (x, y, z)x ----------(29)
However using (1) and C(z, x, x, x) = 0, we have
(x, x, z)x = (z, x, x)x (by (1))
= - z(x, x, x)
= 0
Since (x, x, x) = 0, we have (x, x, z)x = 0
Linearization of this gives
(x, y, z)x + (y, x, z)x + (x, x, z)y $\equiv$ 0
or (x, y, z)x $\equiv$ - (y, x, z)x – (x, x, z)y -----(30)
Using (29), (25) and (26) in (30), we get

$$-\frac{k}{k+1}\alpha + \beta = \frac{\alpha}{k} - \frac{k}{k+1}\beta$$
$$\frac{\alpha}{k} + \frac{k}{k+1}\alpha = \beta + \frac{k}{k+1}\beta$$
$$\left(\frac{k^2+k+1}{k(k+1)}\right)\alpha = \left(\frac{2K+1}{k+1}\right)\beta$$

Using (25) and (26) to substitute for $\frac{\alpha}{k+1}$ *and* $\frac{\beta}{k+1}$ in the above equation gives
$\left(\frac{k^2+k+1}{k}\right)\left(-\frac{1}{k}\right)$ x(x, y, z) $\equiv$ (2k+1) $\left(-\frac{1}{k}\right)$ x(x, z, y)
⇨ (k^2+k+1) x(x, y, z) $\equiv$ ($2k^2$+k) x(x, z, y) -----(31)
Linearizing (31), we obtain
(k^2+k+1) (w(x, y, z) + x(w, y, z)) $\equiv$ ($2k^2$+k) (w(x, z, y) + x(w, z, y)) -----(32)
By substituting w=u=(a, b, c) in (32) and using (11), we get
(k^2+k+1) x(u, y, z) $\equiv$ ($2k^2$+k) x(u, z, y) -----(33)
Linearizing (24) we have (u, z, y) = - (u, y, z)
Using this in (33), we obtain
(k^2+k+1) x(u, y, z) $\equiv$ - ($2k^2$+k) x(u, y, z)

$(3k^2$+2k+1) x(u, y, z) = 0
Thus if $(3k^2$+2k+1) $\neq 0$, we have x(u, y, z) ≡ 0
or [x(u, y, z), n] = 0 for all n ϵ N
Thus (u, y, z) ϵ T. Since T = 0, we have (u, y, z) = 0
Similarly, ($\frac{k^2+k+1}{k(k+1)}$) α ≡ ($\frac{2k+1}{k+1}$) β also yields
(k^2+k+1) (y, z, x)x ≡ $(2k^2 + k)$ (z, y, x)x
Linearizing the above equation, we get
(k^2+k+1) ((y, z, x)w + (y, z, w)x) = $(2k^2 + k)$ ((z, y, x)w + (z, y, w)x)
Putting w=u=(a, b, c) in above and using (11), using (z, y, x)u = 0 and (y, z, x)u = 0, we ge
(k^2+k+1) (y, z, u)x ≡ $(2k^2 + k)$ (z, y, u)x
linearizing (24), we have (z, y, u) = - (y, z, u)
using this in the previous equ, we obtain
(k^2+k+1) (y, z, u)x ≡ - $(2k^2 + k)$ (y, z, u)x
⇨ $(3k^2$+2k+1) (y, z, u)x = 0
Thus if $(3k^2$+2k+1) $\neq 0$, we have (y, z, u)x ≡ 0
or [(y, z, u), n] = 0 for all n ϵ N
Using C (x, y, z, u) = 0 and (x, y, z)u = 0
x(y, z, u) = 0 or (x(y, z, u), n) = 0 for all n ϵ N.
Thus (y, z, u) ϵ T. Since T = 0 we have (y,z,u) = 0
Now we have both (y,z,u) = 0 and (u,y,z) = 0.
Using these two equations in (3) we get (z,u,y) = 0
Now we are in the situation where all associators are in the nucleus.
i.e., (R, R, R) Ç N.
we use result in [2] to conclude that R must be associative.

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